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Existence and Characterization of Optimal Locations

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Abstract. A general model for optimal location problems is given and the existence of solutions is proved under practical conditions. Conditions that all possible solutions must satisfy are given; these conditions form the basis of a method of finding solutions.

Key words: Optimal location, Existence

1. Introduction

The problem of optimally locating a facility relative to existing facilities has a long history. In particular finding a site to minimize distances relative to three other existing locations goes back at least as far as Fermat in the 17th century. The modern interest stems from the work of Weber in 1909. More recent formulations are found in the operations research literature and in the interdisciplinary area of regional science. ReVelle and Laporte (1996) include references to the operations research literature and Thisse (1987) gives the regional science perspective. The recent book of Drezner (1995) on location theory contains a bibliography of over 1200 items.

The objective in this literature is often to locate a distribution, manufacturing, or warehouse facility that minimizes shipping costs from either markets or raw material centers or both, subject to given demands or prices. The distribution of existing facilities is either discrete or given by simple distributions such as the uniform distribution. The problem of locating a facility relative to area demand with a continuous distribution is also treated in the literature (see Drezner and Weslowsky, 1980).

In much of this literature the emphasis is on computational procedures to find the optimal location. The distance functions considered often come from well-known families of metrics such as the l_p-norms. Because of familiar properties (such as convexity) of many of the metrics considered, existence of a minimizing location may be automatic. Some authors do provide existence theorems *per se* (see, for example, Carrizosa, 1995, and Cuesta-Albertos, 1984), but with an accompanying convexity assumption on the distance function. However there are functions having economic import that are concave or even discontinuous. (It is not unusual that the per unit cost of transportation increases with distance, but at a decreasing rate, thus giving a concave function. A discontinuous transportation cost function can be modeled on the postage function).

To try to unify many of the problems, in this paper we provide a general theorem (Theorem 1 in Section 2) on existence of optimal locations with only general hypotheses on the density and cost functions and without assuming convexity. Section 3 contains a mathematical characterization of the solution of a broadly applicable location problem (Theorem 3) as well as several examples of applicable cost functions. Section 4 contains a collection of examples using different densities, cost functions, and dimensions showing how Theorem 3 is applied to find the optimal locations (or their approximations).

The original motivation for this paper is the economic problems associated with locating waste management facilities in cities (see Highfill et al. 1997), although ultimate applications are far less limiting. In this paper, one or more facilities are to be located so as to provide a service to a region at a minimal cost. The facility may have more than one branch; examples include libraries or post offices of a community, shopping centers in a region, and recycling centers of a municipality. The level of service demanded is modeled by a density function over the region. Often this density function will represent the population density but it may be more generally described as the demand for services.

In choosing an optimal facility location, we minimize the transportation cost, which can include any costs that vary with distance. For a library or a post office, this cost may be the total distance travelled by patrons. For a recycling center, this may be the total cost of hauling the waste. In our formulation, the cost is also allowed to depend on the destination. Of course transportation cost is only one of the factors in determining facility locations. Other costs, including environmental protection, facility construction, and maintenance costs are important; however, these concerns can be considered as fixed costs and incorporated as constraints while minimizing the transportation cost. Although a facility will occupy some region in space, the model discussed here will assume the locations are points. Unless the ‘population’ being served by the facilities is highly concentrated on a few small, nearby regions, the problem of overlapping facilities should not occur. Mild conditions in Theorem 2 will show that the optimal locations will be distinct points although their separation, of course, depends on the density of the population.

In addition to location theory, the problem discussed here has features in common with the problem of mass transport, sometimes known as the Monge–Kantorovich problem. Rachev (1985) has written a survey paper on this problem including an extensive set of references. In this problem, two measures and a cost function are given. The problem is to choose a mapping to move the mass associated with the first measure and distribute it according to the second measure, while minimizing the total cost of the transportation. The recent paper of Gangbo and McCann (1996) provides existence and uniqueness results for both the case of convex cost functions as well as concave cost functions. The introduction to their paper contains a useful guide to the vast literature on this problem. After stating our problem, we will be more explicit about its contact with the mass transport problem as formulated by Gangbo and McCann. Finally, the authors wish to thank the referees for several valuable comments.

2. Existence of optimal locations

Suppose μ, a regular Borel measure, represents the distribution of a population (or mass) on $R^m (m \geqslant 1)$. Suppose $\_1, \ldots, z_n$ are the locations of the facilities and $c_1(x, z_1), \ldots, c_n(x, z_n)$ are the costs for a unit population at x to use a facility located at $z_1, \ldots, z_n$, respectively. (In practice, it may be the case that there is only one cost function $c(x, z)$ (i.e., $c_i = c$) or that the various cost functions are weighted versions of a single cost function ($c_i(x, z_i) = w_i c(x, z_i)$).) Then R^m is partitioned into n (disjoint) regions $D_1, \ldots, D_n$ and the population in D_i is supposed to use the facility at z_i, $i = 1, \ldots, n$. The total cost can be expressed as an integral

$$F = \sum_{i=1}^{n} \int_{D_i} c_i(x, z_i)\, d\mu(x). \tag{1}$$

The problem is, for a given positive integer n, to find locations $z_1, \ldots, z_n$ and a (disjoint) partition $D_1, \ldots, D_n$ of R^m such that the total cost F is a minimum among all possible $z_1, \ldots, z_n$ and $D_1, \ldots, D_n$.

With our problem formally stated, we can show the connection between our problem and the Monge problem in Gangbo and McCann (1996). The primary difference in the two problems is in what is sought. In the Monge problem, a transport function is sought to minimize total cost between two masses with given densities. In our problem, the locations z_i and their associated regions D_i can be thought of as generating a discrete measure corresponding to the target measure in the Monge problem. Our problem asks for the cost-minimizing locations with the resulting transport function being simply: map all points in D_i to z_i. Gangbo and McCann do allow a finite discrete measure in the range space, but again it is fixed and their objective is to find the transport function. In terms of applications, the problem in this paper is one of locating new facilities for existing markets, while the Monge problem is, roughly, to determine the spatial markets for fixed factories.

Note that existence of minimizing locations is not always assured. For example, let μ be the measure with support $\mathrm{spt}(\mu) = \{0, 1\}$ and values $\mu\{0\} = \mu\{1\} = 1$. Consider the cost function $c(x, z) = \lfloor\sqrt{|x - z|}\rfloor + \sqrt{|x - z|}$, where $\lfloor t \rfloor$ is the floor function, i.e., the greatest integer less than or equal to t. If we try to find the optimal location for a single facility located at $z \in R$, the total cost function is

$$F(z) = (\lfloor\sqrt{|0 - z|}\rfloor + \sqrt{|0 - z|}) + (\lfloor\sqrt{|1 - z|}\rfloor + \sqrt{|1 - z|}).$$

A simple computation shows that $\inf_z F(z) = 1$, and, in fact, $\lim F(z) = 1$ as z approaches either 0 or 1 from within the interval (0, 1). However the function has jump discontinuities at $z = 0$ and 1 and the infimum is not obtained for any z. A modification of this example is discussed below which shows that even if the minimum transportation cost exists, it need not be unique.

In light of the assumptions that are about to be introduced, it should be noted that the cost function c is upper semicontinuous in z. This shows that the condition of lower semicontinuity discussed below cannot be dismissed altogether. However it is easy to see that if the cost function above is replaced with the function $c(x, z) = \lfloor |x - z| \rfloor$, which is also upper semicontinuous, the minimum total cost is achieved for any $z \in [0, 1]$. So, as will be seen in Theorem 1, lower semicontinuity is part of the sufficient conditions, but is not necessary for achieving the minimum.

There are several topologically equivalent norms on R^m. Fix one of these norms and denote it by $\|x\|$. We need some hypotheses on the cost and density functions.

ASSUMPTIONS.

(1) Suppose n is a given integer and for $i = 1, \dots, n$,
   (i) $c_i(x, z) : R^m \times R^m \to [0, \infty)$ is lower semicontinuous in z and measurable in x;
   (ii) there exists $c_{i0} \in (0, \infty]$ such that for any $r > 0$, $\sup_{\|x\|, \|z\| \leqslant r} c(x, z) < c_{i0}$;
   (iii) $\lim_{\|z\| \to \infty} c(x, z) = c_{i0}$ uniformly for $\|x\| \leqslant r$.

(2) μ has finite mass; and $\int_{R^m} c_i(x, z)\, d\mu(x) < \infty$ for all $z \in R^m$.

Assumption (1.iii) enables us to define $c_i(x, \infty) = c_{i0}$ for $i = 1, \dots, n$. For given locations $z_1, \dots, z_n$, there is a best way to partition R^m into subregions $D_1, \dots, D_n$ so that the cost is a minimum, compared with all other partitions of R^m with the given locations $z_1, \dots, z_n$. Indeed, for $i = 1, \dots, n$, define

$$\begin{aligned} D_1 &= \{x \in R^m : c_1(x, z_1) \leqslant c_j(x, z_j) \text{ for all } j > 1\}, \\ D_2 &= \{x \in R^m \backslash D_1 : c_2(x, z_2) \leqslant c_j(x, z_j) \text{ for all } j > 2\}, \\ &\dots \\ D_n &= R^m \backslash (D_1 \cup \dots \cup D_{n-1}) \end{aligned} \tag{2}$$

that is, D_i consists of all x where the population has lower cost to use the facility at z_i than at all other z_j. When the cost functions are a function of the Euclidean distance, the partition $D_1, \dots, D_n$ is the so-called Voronoi diagram which is used by Suzuki and Okabe (1995) to approximate optimal locations. With these choices of $D_1, \dots, D_n$, the cost is a function of $z_1, \dots, z_n$ alone, which we denote by

$$F(z_1, \dots, z_n) = \sum_{i=1}^{n} \int_{D_i} c_i(x, z_i)\, d\mu = \int_{R^m} C(x, z_1, \dots, z_n)\, d\mu \tag{3}$$

where $C(x, z_1, \dots, z_n) = \min\{c_i(x, z_i), i = 1, \dots, n\}$.

It will be used repeatedly that the functions C and F are lower semicontinuous as functions of the z_i's.

LEMMA 1. *$C(x, z_1, \ldots, z_n)$ is measurable in x and lower semicontinuous in $z_1, \ldots, z_n$.*

Proof. From (3) and the Assumption (1.i) we see that $C(x, z_1, \ldots, z_n)$ is measurable in x. Assume that $z_1^k, \ldots, z_n^k$ is a sequence converging to $z_1^0, \ldots, z_n^0$ (some of $z_1^0, \ldots, z_n^0$ could be points at infinity) and $C(x, z_1^k, \ldots, z_n^k) \to l(x)$ as $k \to \infty$. By passing to a subsequence, we may assume that $c_i(x, z_i^k) \to l(x)$ for some particular $i \in \{1, \ldots, n\}$. Because $c_i(x, z)$ is lower semicontinuous in z, $c_i(x, z_i^0) \leqslant l(x)$. Therefore,

$$C(x, z_1^0, \ldots, z_n^0) \leqslant c_i(x, z_i^0) \leqslant l(x) = \lim_{k\to\infty} C(x, z_1^k, \ldots, z_n^k). \tag{4}$$

So $C(x, z_1, \ldots, z_n)$ is lower semicontinuous. □

LEMMA 2. *The function $F(z_1, \ldots, z_n)$ is lower semicontinuous in $z_1, \ldots, z_n$.*

Proof. Again if $z_1^k, \ldots, z_n^k$ is a sequence converging to $z_1^0, \ldots, z_n^0$ and $F(z_1^k, \ldots, z_n^k) \to F_0$ as $k \to \infty$, then by the lower semicontinuity of C,

$$C(x, z_1^0, \ldots, z_n^0) \leqslant l(x) \equiv \varliminf_{k\to\infty} C(x, z_1^k, \ldots, z_n^k),$$

which implies that

$$F(z_1^0, \ldots, z_n^0) = \int_{R^m} C(x, z_1^0, \ldots, z_n^0)\, d\mu \leqslant \int_{R^m} l(x)\, d\mu.$$

By Fatou's lemma, applied to $C(x, z_1^k, \ldots, z_n^k)$,

$$\int_{R^m} l(x) d\mu \leqslant \varliminf_{k\to\infty} \int_{R^m} C(x, z_1^k, \ldots, z_n^k)\, d\mu = \lim_{k\to\infty} F(z_1^k, \ldots, z_n^k) = F_0.$$

So $F(z_1^0, \ldots, z_n^0) \leqslant F_0$, that is, F is lower semicontinuous. □

It is well known that a lower semicontinuous function on a compact set achieves its minimum. The following theorem shows that F achieves its minimum without the compactness hypothesis.

THEOREM 1. *Under Assumptions (1)–(2), there are locations $z_1, \ldots, z_n$ such that F is a minimum among all possible locations.*

Proof. For simplicity, we assume that μ has total mass 1. Let $c_0 = \min\{c_{10}, \ldots, c_{n0}\}$. It follows from Assumption (1) that for all $z_1, \ldots, z_n$,

$$F(z_1, \ldots, z_n) < c_0. \tag{5}$$

This is because the opposite, that $F(z_1, \ldots, z_n) \geqslant c_0$, of (5) would imply that

$$\int_{R^m} [C(x, z_1, \ldots, z_n) - c_0]\, d\mu \geqslant 0.$$

However, $C(x, z_1, \dots, z_n) - c_0$ is strictly less than zero; this contradicts the assumption $\int_{R^m} d\mu = 1$. So (5) holds.

The greatest lower bound of F corresponding to all possible $z_1, \dots, z_n$, denoted by $F_{\min}$, exists, of course, and the previous paragraph implies

$$F_{\min} < c_0. \tag{6}$$

Take a minimizing sequence $\{z_1^k, \dots, z_n^k\}_{k=1}^{\infty}$ such that $F(z_1^k, \dots, z_n^k) \to F_{\min}$ as $k \to \infty$. If the minimizing sequence converges, then the limit is the desired minimizer because of the lower semicontinuity of F, as shown in Lemma 2 above. In case the sequence does not converge, we need a (subsequential) limit for at least one of the z_i^k's and for that we prove the following assertion.

ASSERTION. *There is an $i \in \{1, \dots, n\}$ such that $\{z_i^k\}$ contains a bounded subsequence.*

Proof. Suppose instead that $\|z_i^k\| \to \infty$ for all $i \in \{1, \dots, n\}$ as $k \to \infty$. It follows that for any $r > 0$, we have

$$\begin{aligned} F(z_1^k, \dots, z_n^k) &= \int_{R^m} C(x, z_1^k, \dots, z_n^k)\, d\mu \\ &\geqslant \int_{\|x\| \leqslant r} C(x, z_1^k, \dots, z_n^k)\, d\mu \\ &\geqslant \inf_{\|x\| \leqslant r} C(x, z_1^k, \dots, z_n^k) \left(\int_{\|x\| \leqslant r} d\mu \right). \end{aligned} \tag{7}$$

There are two cases.

Case (1): $c_0 = \infty$. Then choose an r such that $\int_{\|x\| \leqslant r} d\mu > 0$. Assumption (1.ii) implies that $\inf_{\|x\| \leqslant r} C(x, z_1^k, \dots, z_n^k) \to c_0 = \infty$ as $k \to \infty$. Therefore, (7) implies that $\int_{R^m} C(x, z_1^k, \dots, z_n^k)\, d\mu \to \infty$. This contradicts the fact that $F_{\min} < \infty$.

Case (2): $c_0 < \infty$. Then we can

(i) choose an $\epsilon > 0$ such that $(1 - \epsilon)(c_0 - \epsilon) > F_{\min}$; this is possible by (6).

(ii) choose an $r > 0$ such that $\int_{\|x\| \leqslant r} d\mu > 1 - \epsilon$; this can be done because $\int_{R^m} d\mu = 1$.

(iii) for ϵ and r in (i)–(ii), choose a K such that $C(x, z_1^k, \dots, z_n^k) > c_0 - \epsilon$ for all $k \geqslant K$ and x with $\|x\| \leqslant r$. This is guaranteed by Assumption (1.iii).

Then for $k \geqslant K$, (7) implies that

$$\int_{R^m} C(x, z_1^k, \dots, z_n^k)\, d\mu \geqslant (1 - \epsilon)(c_0 - \epsilon)$$

which implies $F_{\min} \geqslant (1 - \epsilon)(c_0 - \epsilon) > F_{\min}$, a contradiction. So the Assertion is proved.

The Assertion says that at least one of the sequences $z_1^k, \dots, z_n^k$ contains a bounded subsequence. By rearranging the order and passing to subsequences if necessary, we may classify the sequences into convergent and divergent: there is an n', $1 \leqslant n' \leqslant n$ such that as $k \to \infty$,

$$z_1^k, \dots, z_{n'}^k \to z_1^0, \dots, z_{n'}^0, \text{ and } \|z_{n'+1}^k\|, \dots, \|z_n^k\| \to \infty$$

We need to further distinguish the indices $\{n'+1, \dots n\}$. By rearranging if necessary, assume that $c_{i0} < \infty$ for $n' < i \leqslant n''$, and $c_{i0} = \infty$ for $i > n''$. So the index set is $\{1, \dots, n\} = \{1, \dots, n', \dots, n'', \dots, n\}$. The z_i^k's have a finite limit for $i < n'$. The terms $c_i(x, z_i^k)$ have a finite limit for $i \leqslant n''$ and an infinite limit for $i > n''$ as $k \to \infty$.

The next part of the proof shows that the only part of the index set that is relevant for the minimizing sequences $\{z_i^k\}$ is $i = 1, \dots, n'$. We first show that only the first n'' indices matter. From definition (3), we know that since C is a minimum of $c_i(x, z_i^k)$'s, when k is sufficiently large, the index i that gives this minimum will be less than or equal to n'' since $c_{i0} = \infty$ for $i > n''$. Thus

$$C(x, z_1^k, \dots, z_n^k) = C(x, z_1^k, \dots, z_{n''}^k)$$

for $\|x\| \leqslant r$. (We are abusing notation here slightly; even though the right-hand side is only a function of $n''+1$ variables, the same function notation C is used; it is defined to be $C(x, z_1^k, \dots, z_{n''}^k) = \min\{(c_i(x, z_i^k), i = 1, \dots, n''\}.$)

Finally we show that the minimizing sequences $\{z_i^k\}$ with indices $i = n' + 1, \dots, n''$ will also have no effect on the minimization of F. (These are all points at infinity and so are undesirable for optimal locations.) Let $r > 0$ be a fixed number such that $\|z_1^k\|, \dots, \|z_{n'}^k\| \leqslant r$. By Assumption 1.iii, for all $\|x\| \leqslant r$, we know that $c_i(x, z_i^k) < c_{i0}$ for $i = 1, \dots, n'$ and all k's, while for $i = n'+1, \dots, n$ we have $c_i(x, z_i^k) \to c_{i0}$ (uniformly in x) as $k \to \infty$. By the lower semi-continuity of $F(z_1, \dots, z_{n''})$,

$$\begin{aligned}
&\int_{\|x\| \leqslant r} \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1,0}, \dots, c_{n'',0}\}\, d\mu \\
&\leqslant \lim_{k\to\infty} \int_{\|x\| \leqslant r} C(x, z_1^k, \dots, z_{n''}^k)\, d\mu \\
&= \lim_{k\to\infty} \int_{\|x\| \leqslant r} C(x, z_1^k, \dots, z_n^k)\, d\mu \\
&\leqslant \lim_{k\to\infty} \int_{R^m} C(x, z_1^k, \dots, z_n^k)\, d\mu \\
&= F_{\min}.
\end{aligned}$$

Since r is arbitrary, taking the limit as $r \to \infty$, we have

$$F\left(z_1^0, \dots, z_{n'}^0, \overbrace{\infty, \dots, \infty}^{n''-n'}\right)$$
$$= \int_{R^m} \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1,0}, \dots, c_{n'',0}\}\, d\mu \leqslant F_{\min}.$$

We can now show that the locations minimizing F are points all of whose coordinates are finite. That is, $F_{\min} = F(z_1^0, \dots, z_{n'}^0)$. To show this we prove that except on a set of measure zero, $\min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1,0}, \dots, c_{n'',0}\} = \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0)\}$. Let S be the set of the points x such that

$$\min\{c_i(x, z_i^0), i \leqslant n'\} \leqslant \min\{c_{j0}, j = n' + 1, \dots, n''\}.$$

We show S has full measure: $\mu[S] = 1$. For otherwise suppose that $\mu[S'] > 0$. Consider locations the $z_1^0, \dots, z_{n'}^0$ with any other finite points $z_{n'+1}^*, \dots, z_{n''}^*$. Then Assumption 1.ii implies that

$$\int_{S'} [\min\{c_{n'+1}(x, z_{n'+1}^*), \dots, c_{n''}(x, z_{n''}^*)\}$$
$$- \min\{c_{j0}, j = n' + 1, \dots, n''\}]\, d\mu < 0$$

because the integrand is everywhere negative on S'. Therefore

$$\int_{R^m} \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1}(x, z_{n'+1}^*), \dots, c_{n''}(x, z_{n''}^*)\}\, d\mu$$
$$\leqslant \int_S \min\{c_1(x_1, z_1^0), \dots, c_{n'}(x, z_{n'}^0)\}\, d\mu$$
$$+ \int_{S'} \min\{c_{n'+1}(x, z_{n'+1}^*), \dots, c_{n''}(x, z_{n''}^*)\}\, d\mu$$
$$< \int_S \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0)\}\, d\mu + \int_{S'} \min\{c_{n'+1,0}, \dots, c_{n'',0}\}\, d\mu$$
$$= \int_{R^m} \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1,0}, \dots, c_{n'',0}\}\, d\mu$$
$$\leqslant F_{\min}$$

which contradicts the definition of $F_{\min}$. Therefore for μ- almost all x,

$$\min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0), c_{n'+1,0}, \dots, c_{n'',0}\}$$
$$= \min\{c_1(x, z_1^0), \dots, c_{n'}(x, z_{n'}^0)\} = C(x, z_1^0, \dots, z_{n'}^0)$$

So $F(z_1^0, \dots, z_{n'}^0, \overbrace{\infty, \dots, \infty}^{n''-n'}) = F(z_1^0, \dots, z_{n'}^0)$ and $z_1^0, \dots, z_{n'}^0$ gives a minimum for F. $\square$

REMARK 1. If all the cost functions $c_i(x, z)$ are the same, proof can be simplified considerably. In particular, because in this case all the c_{i0}'s are equal to c_0, so the minimizing sequence can be divided into only two groups. The sequences $\{z_i^k\}_{k=1}^{\infty}$ such that $\|z_i^k\| \to \infty$, either all satisfy $c_0 < \infty$ or all satisfy $c_0 = \infty$.

REMARK 2. The original problem was to find n points in R^m to minimize the transportation cost. Yet the solution above may yield fewer than n points. The interpretation is that the other $n - n'$ points are "duplicates" of the ones found. That is, if $n' \neq n$, the minimizing locations are not all distinct. A simple condition guaranteeing distinct locations will be given in Theorem 2.

Examples of per unit cost functions. Denote by $x = (x^{(1)}, \ldots, x^{(m)})$ and $z = (z^{(1)}, \ldots, z^{(m)})$ the coordinates of $x, z \in R^m$. Theorem 1 applies to the following cost functions.

(a) Let p, q be two positive numbers,

$$c(x, z) = \left(\sum_{j=1}^{m} |x^{(j)} - z^{(j)}|^p \right)^q .$$

More specific examples included in (a) are

$$\text{Manhattan (i.e. } l_1\text{) metric: } c(x, z) = \sum_{j=1}^{m} |x^{(j)} - z^{(j)}|$$

$$l_p \text{ metric: } c(x, z) = \|x - z\|_p = \left(\sum_{j=1}^{m} |x^{(j)} - z^{(j)}|^p \right)^{1/p}, \; p > 1.$$

$$\text{Concave metric: } c(x, z) = \left(\sum_{j=1}^{m} |x^{(j)} - z^{(j)}| \right)^q, \; 0 < q < 1.$$

(b) Let $\rho(t)$ be a positive and strictly increasing function and $c(x, z)$ be one of the functions in (a). Then $\rho(c(x, z))$ also satisfies the Assumptions. Typical examples of such functions $\rho(t)$ useful for transportation problems include $\ln(1 + t)$ and $\frac{t}{1+t}$. An example of a discontinuous and nondecreasing function that may occur as ρ is the postage stamp function $p(t)$. (This is essentially the floor function with the values at the points of discontinuity changed so as to make the function continuous from the left.) Note that this is a lower semicontinuous function. Then $p(\|x - z\|)$ satisfies the Assumptions.

(c) Suppose $c_1(x, z)$ and $c_2(x, z)$ are cost functions satisfying the Assumptions. Let $\lambda > 0$ be a constant and $w \in R^m$ is fixed, then

$$c'(x, z) \equiv c_1(x, z) + \lambda c_2(z, w)$$

also satisfies the Assumptions. The motivation for this example is a problem in municipal waste management, as discussed next. See also Highfill et al. (1997).

(d) Suppose a city, situated in a planar region D in R^2, has m landfills $w_1, \ldots, w_m$ and wishes to locate n recycling centers $z_1, \ldots, z_n$. A common waste management plan is as follows. The city is divided into n subdivisions $D_1, \ldots, D_n$. The waste in region D_i is transported to the nearest (in travel cost) recycling center z_i, where the recyclables are sorted and taken away by a recycler at no further cost to the city. The non-recyclable part of the waste is then taken to a landfill by the city. Let $f(x)$ be the density of the waste that must be disposed of by the city. (A substitute for the waste density could be a suitable fraction of the population density.) Let $c_1(x, z)$ be unit the transportation cost of the first stage of waste collection (i.e., households to recycling center). For $i = 1, \ldots, n$, the waste in D_i is transported to z_i at total cost $\int_{D_i} f(x)c_1(x, z_i)dx$. At each recycling center, γ is the fraction of the waste that is recycled. The non- recyclable part, $(1-\gamma)\int_{D_i} f(x)dx$, is transported to the nearest landfill at cost $(1-\gamma)c_2(z_i)\int_{D_i} f(x)dx$ where $c_2(z_i) = \min\{c_1(z_i, w_1), \ldots, c_1(z_i, w_m)\}$ is the unit cost from z_i to the nearest landfill. (The model can be generalized slightly by assuming that the per unit cost in the second stage is different from that in the first. This would represent, for example, cost savings of bulk shipment of waste from the recycling center to the landfill.) Therefore, the total transportation cost is the sum of the two costs over all the subdivisions:

$$\begin{aligned} F &= \sum_{i=1}^{n} \left[\int_{D_i} f(x)c_1(x, z_i)\, dx + (1-\gamma)c_2(z_i) \int_{D_i} f(x)\, dx \right] \\ &= \sum_{i=1}^{n} \int_{D_i} f(x)c(x, z_i)\, dx \end{aligned}$$

where $c(x, z_i) = c_1(x, z_i) + (1-\gamma)c_2(z_i)$.

3. Properties and characterization of optimal locations

Returning to the original problem, it should not be surprising that “more is better” when considering the number of locations. Increasing the number of locations will decrease the total distance (and hence cost) travelled. We have the following theorem showing when the locations $z_1, \ldots, z_n$ are all distinct. The theorem is restricted to a single cost function (i.e., $c_i = c$ for $i = 1, \ldots, n$).

THEOREM 2. *Suppose Spt(μ) contains at least n points. If the cost c is continuous and*

$$c(x, x) < c(x, z) \tag{8}$$

for all $x \neq z$, then the optimal locations $z_1, \dots, z_n$ are mutually distinct.

Proof. Suppose $z_1, \dots, z_n$ are optimal locations but some of them are equal. Then we show by adding one more location, the total cost decreases.

From Assumption (2), there exists a point $z_0 \in R^m$, $z_0 \neq z_1, \dots, z_n$, such that $\int_{B(z_0,r)} d\mu(x) > 0$ for all $r > 0$, where $B(z_0, r)$ is a ball of radius r. Condition (8) implies that $c(z_0, z_0) < c(z_0, z_i)$, $i = 1, \dots, n$. Because $c(x, z)$ is continuous, there is a number $r > 0$ such that $c(x, z_0) < c(x, z_i)$, $i = 1, \dots, n$ for all $x \in B(z_0, r)$. Therefore, for $x \in B(z_0, r)$, $C(x, z_1, \dots, z_n) > C(x, z_0, z_1, \dots, z_n) = c(x, z_0)$. By the choice of z_0,

$$\int_{B(z_0,r)} C(x, z_1, \dots, z_n)\, d\mu(x) > \int_{B(z_0,r)} C(x, z_0, z_1, \dots, z_n)\, d\mu(x).$$

As a result,

$$\begin{aligned} F(z_0, z_1, \dots, z_n) &= \int_{R^m} C(x, z_0, z_1, \dots, z_n)\, d\mu \\ &< \int_{R^m} C(x, z_1, \dots, z_n)\, d\mu = F(z_1, \dots, z_n). \end{aligned}$$

So if, for example, $z_1 = z_2$, we can replace z_1 with z_0 and find that $F(z_0, z_2, \dots, z_n) < F(z_1, z_2, \dots, z_n)$ contradicting optimality of the points $z_1, z_2, \dots, z_n$. □

As mentioned above, if facilities are located at $z_1, \dots, z_n$ (perhaps not optimally), then there is a best partition $D_1, \dots, D_n$ of R^m giving smallest cost for *this* choice of facility locations. The cost is

$$F(z_1, \dots, z_n) = \sum_{i=1}^{n} \int_{D_i} c_i(x, z_i)\, d\mu(x).$$

For $i = 1, \dots, n$, we consider the following function

$$F_i(z; z_1, \dots, z_n) = \int_{D_i} c_i(x, z)\, d\mu(x).$$

This is the cost associated with the region D_i with a facility located at z; the cost also depends on $z_1, \dots, z_n$ because D_i is defined using $z_1, \dots, z_n$ and the cost function c_i. If $z = z_i$, this is exactly the i-th term in $F(z_1, \dots, z_n)$ and

$$F(z_1, \dots, z_n) = \sum_{i=1}^{n} F_i(z_i; z_1, \dots, z_n).$$

Of course z_i may not minimize $F_i(z; z_1, \dots, z_n)$ if the z_k's are chosen arbitrarily; i.e., z_i may not be an optimal facility location for D_i. On the other hand, if $z_1, \dots, z_n$ are optimal facility locations, then each z_i minimizes the corresponding F_i.

THEOREM 3. *If $z_1, \dots, z_n$ minimize F, then z_i minimizes $F_i(z; z_1, \dots, z_n)$ for $i = 1, \dots, n$, respectively.*

This gives a collection of necessary conditions for a minimum, which can be written as a system of equations. From this system, all possible optimal locations can be found. Some examples will be found in Section 4.

Proof. Suppose $x_1, \dots, x_n$ are arbitrary locations of the facility and $E_1, \dots, E_n$ is an arbitrary partition. If $z_1, \dots, z_n$ are optimal locations (and D_i the corresponding regions), then

$$\sum_{i=1}^{n} \int_{D_i} c_i(x, z_i) d\mu(x) \leqslant \sum_{i=1}^{n} \int_{E_i} c_i(x, x_i)\, d\mu(x).$$

Fix i and take $E_j = D_j$ for all j, and $x_j = z_j$ for all $j \neq i$, then it follows that

$$\int_{D_i} c_i(x, z_i)\, d\mu(x) \leqslant \int_{D_i} c_i(x, x_i)\, d\mu(x).$$

So z_i is a minimum of $F_i(z; z_1, \dots, z_n) = \int_{D_i} c_i(x, z)\, d\mu$, for $i = 1, \dots, n$. □

4. Examples

We conclude the paper with a collection of examples. Although the main contribution of the paper is to establish a general existence theorem (Theorem 1), Theorem 3 gives a characterization of the optimal locations that can be used to help find the desired locations. The following examples are designed to show some of the features of the problem and possible solutions as suggested by Theorem 3. Examples 1–3 show the effect of changing the cost function and possible nonuniqueness of optimizing locations. These examples use necessary conditions given by Theorem 3, but in situations that are simple enough to permit explicit computation. Examples 5–8 show how to use Theorem 3 to compute locations for some density functions that are not merely uniform or atomic. For related works including computations in a spirit similar to ours, we refer to Braid (1996), Cuesta-Albertos (1984), Drezner (1994), and Drezner's book (1995) both for its collection of papers and for its bibliography.

EXAMPLE 1 (one location, one dimension: $n = 1, m = 1$). Suppose we are interested in one optimal location. In this case, the total cost is $F(z) = \int_{R^m} c(x, z)\, d\mu$. We have the following results

(a) If $c(x, z) = h(\|x - z\|)$ and h is strictly convex, then a computation shows that F is strictly convex and thus the minimum is unique.

A typical example is $c(x, z) = \|x - z\|_2^2 = \sum_{j=1}^{m} |x^{(j)} - z^{(j)}|^2$. In this case, the optimal location is precisely the *center of mass* with distribution μ.

(b) If $c(x,z) = \|x - z\| = \sum_{j=1}^{m} |x^{(j)} - z^{(j)}|$, then the optimal location is the *median* of the mass.

(c) If $c(x,z) = h(\|x - z\|)$ and h is strictly concave, then the minimum may not be unique. For example, consider $h(t) = \sqrt{t}$ and the atomic measure $\mu\{0\} = \mu\{1\} = 0.5$. In this case the total cost function for locating a facility at z is $F(z) = 0.5(\sqrt{|z|} + \sqrt{|z-1|})$. The minimum occurs at $z = 0$ and $z = 1$ where F has the value 0.5.

(d) Perhaps the simplest example in which a minimum is not unique is found by letting the mass be concentrated at two points as in the previous example. Let $n = m = 1$ and define $\mu\{0\} = \mu\{1\} = 0.5$ and $\mathrm{Spt}(\mu) = \{0, 1\}$ and let $c(x,z) = |x - z|$. Then any $z \in (0,1)$ will give $F(z) = 1$. The example can be extended to two dimensions in several ways. Define $\mu\{(1,0)\} = \mu\{(0,1)\} = \mu\{(-1,0)\} = \mu\{(0,-1)\}$. Use the l_1-metric, and find that the location minimizing total cost is not unique.

EXAMPLE 2 (two locations, one dimension: $n = 2, m = 1$, different cost functions). In this case,

$$F = \int_{D_1} c_1(x, z_1)\, d\mu + \int_{D_2} c_2(x, z_2)\, d\mu$$

where z_1 and z_2 are the locations and D_1 and D_2 form the partition of R. Let μ be ordinary Lebesgue measure (i.e. the population is uniformly distributed) on the interval $[-1, 1]$. Let $c_1(x,z) = |x - z|$ and $c_2(x,z) = 2|x - z|$. Assume that $z_1 < z_2$. The division between regions D_1 and D_2 occurs at the value for which the two cost functions are equal; this is $m = \frac{z_1 + 2z_2}{3}$. The optimal locations are easily found by minimizing the function of two variables

$$F(z_1, z_2) = \int_{-1}^{\frac{z_1+2z_2}{3}} |x - z_1|\, dx + \int_{\frac{z_1+2z_2}{3}}^{1} 2|x - z_2|\, dx.$$

But to show the use of Theorem 3, we consider the two pieces separately. Slightly modifying the notation of Theorem 3 for clarity, write

$$F(z; z_1, z_2) = \int_{-1}^{\frac{z_1+2z_2}{3}} |x - z|\, dx \text{ and } F_2(w; z_1, z_2) = \int_{\frac{z_1+2z_2}{3}}^{1} 2|x - w|\, dx.$$

The first order conditions for these two functions are

$$\frac{\partial F_1}{\partial z} = 0 \text{ yields} \quad z = \frac{z_1 + 2z_2}{6} - \tfrac{1}{2}$$
$$\frac{\partial F_2}{\partial w} = 0 \text{ yields} \quad z = \frac{z_1 + 2z_2}{6} + \tfrac{1}{2}.$$

Theorem 3 says that $z = z_1$ and $w = z_2$. This gives a system of equations in z_1 and z_2 which yields $z_1 = -1/3$ and $z_2 = 2/3$.

EXAMPLE 3 (two locations, one dimension: $n = 2, m = 1$, same cost function). Suppose $z_1 \leqslant z_2$, and $c_i(x, z) = h(|x - z|)$, where h is an increasing function, then the best partitions are

$$D_1 = (-\infty, m),\ D_2 = (m, \infty)$$

where $m = \frac{z_1+z_2}{2}$. So

$$F = \int_{-\infty}^{m} c(x, z_1)\, d\mu + \int_{m}^{\infty} c(x, z_2)\, d\mu.$$

(a) Suppose $c(x, z) = |x - z|^2$ and z_1, z_2 are the optimal locations, then z_1, z_2 must be the centers of the mass in $D_1 = (-\infty, m)$, $D_2 = (m, \infty)$, respectively. That is,

$$z_1 \int_{-\infty}^{m} d\mu = \int_{-\infty}^{m} x\, d\mu \text{ and } z_2 \int_{m}^{\infty} d\mu = \int_{m}^{\infty} x\, d\mu.$$

In fact, these two equations determine all possible optimal locations.
(b) Suppose $c(x, z) = |x - z|$ and z_1, z_2 are the optimal locations, then z_1, z_2 must be the medians of the mass in $D_1 = (-\infty, m)$, $D_2 = (m, \infty)$, respectively. Namely

$$2\int_{-\infty}^{z_1} d\mu = \int_{-\infty}^{m} d\mu,\ 2\int_{z_2}^{\infty} d\mu = \int_{m}^{\infty} d\mu.$$

Again these equations determine all possible optimal locations.

EXAMPLE 4 (two locations, two dimensions: $n = 2$, $m = 2$). For a specific example, suppose that μ is the uniform distribution (i.e. Lebesgue measure) on the unit square D with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$, and suppose we are interested in the optimal locations of two facilities. Given two locations z_1 and z_2, let D_1, D_2 be optimal partitions of D with respect to z_1, z_2. Then the total cost is

$$F = \int_{D_1} c((x^{(1)}, x^{(2)}), z_1)\, dx^{(1)}\, dx^{(2)} + \int_{D_2} c((x^{(1)}, x^{(2)}), z_2)\, dx^{(1)}\, dx^{(2)}$$

where c is the cost function. Examples with specific per unit cost functions follow.
(a) Suppose c is the square of the l_2-norm: $c(x, z) = c[(x^{(1)}, x^{(2)}), (z^{(1)}, z^{(2)})] = (x^{(1)} - z^{(1)})^2 + (x^{(2)} - z^{(2)})^2$. The first order conditions show that D_1, D_2, the regions associated with the desired locations z_1 and z_2, are divided by the perpendicular bisector of the segment between z_1 and z_2. Using Theorem 3, we can restrict attention to one region at a time. It is easy to show that the necessary condition for the optimal locations is that z_1, z_2 are the centers of masses in D_1, D_2, respectively. The task then is to choose the correct line with which to divide the unit square. This reduced the problem to finding the two parameters that describe the optimal line. The first order conditions show that D_1, D_2 are divided by one of the lines $x = 0.5$, $z = 0.5$, $z = x$, or $z = -x$. That is, the optimal division is two rectangles

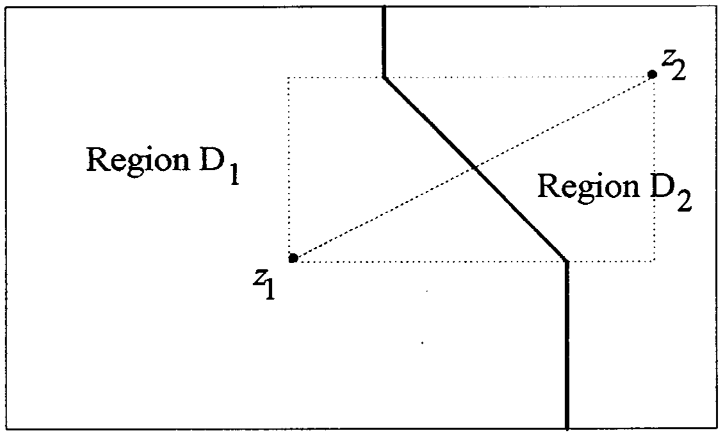

Figure 1.

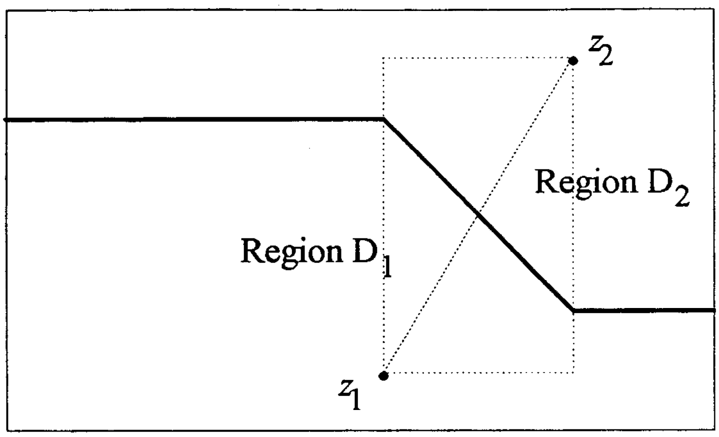

Figure 2.

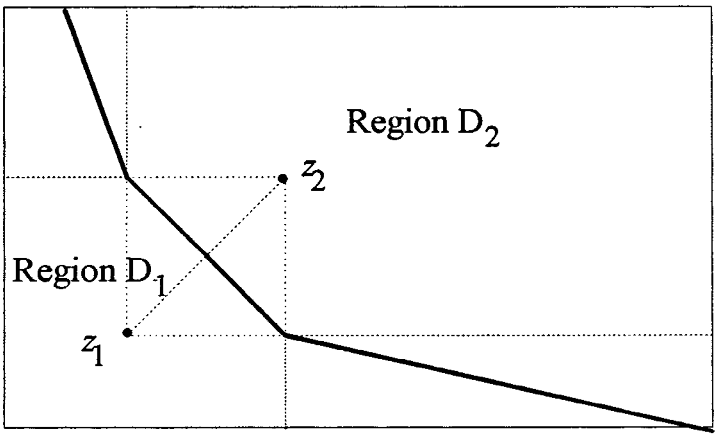

Figure 3.

beside (or on top of) each other or the two 45° triangles. (Because of the rotational symmetry in these solutions, we are really left with two candidates.) By comparing the total costs, we see that the optimal regions are the rectangular ones and the optimal locations are (0.25, 0.5), (0.75, 0.5), or (0.5, 0.25), (0.5, 0.75).
(b) Suppose $c(x, z) = c[(x^{(1)}, x^{(2)}), (z^{(1)}, z^{(2)})] = |x^{(1)} - z^{(1)}| + |x^{(2)} - z^{(2)}|$ (the l_1 or 'Manhattan' metric). Then D_1 and D_2 are (usually) divided by a zigzag boundary, as shown in Figure 1. This example gives a hint at the effects of changing the metric from the one in (a). To provide a few details on the figures, look first at Figure 1. Imagine two locations z_1 and z_2 (not necessarily optimal). To find the boundary between regions D_1 and D_2, first draw a rectangle with corners z_1 and z_2. The middle part of the boundary is a 45° line through the midpoint of the segment between z_1 and z_2. This extends to the sides of the rectangle. The boundary between regions is completed with lines perpendicular to the sides of the rectangle. Another view of this is given in Figure 2. A special case occurs when the rectangle with corners z_1 and z_2 is actually a square, as shown in Figure 3. In this case the middle part of the boundary between D_1 and D_2 is the diagonal of the square. The remaining parts of the boundary are arbitrary – as long as they remain in the dashed rectangles shown in the figure.

Now that the regions have been described for arbitrary locations z_1 and z_2, we can again use Theorem 3: the necessary condition for z_1, z_2 to be the optimal locations of two facilities is that z_1, z_2 are the medians of the masses in D_1, D_2, respectively. It can be shown that the optimal solution in this case is, as in the previous example: D_1, D_2 are divided by $x^{(1)} = 0.5$, or $x^{(2)} = 0.5$, and the optimal locations $(0.25, 0.5)$, $(0.75, 0.5)$ or $(0.5, 0.25)$, $(0.5, 0.75)$.

The rest of the examples all use the cost function $c(x, z) = \|x - z\|_2^2$. As discussed in Examples 2(a) and 3(a), the resulting optimal locations are all centers of mass of the population density in an appropriate region. The number of locations, dimension, and population density are varied. In each example, the population density is given by a function $\rho(x)$. These examples are meant only to illustrate possible uses of the necessary conditions given by Theorem 3. The 'spirit' of the calculation is similar to that in Drezner and Weslowsky (1980) and Drezner and Drezner (1997) in the following sense. Initial location(s) are selected. The first order conditions are constructed and evaluated at these locations producing new locations. In this way a sequence of locations is produced that (under favorable circumstances) converges to the desired optimum. In the cited 1980 paper, the cost function is assumed to be the l^p-norm and the example there uses the l^2-norm. As part of the procedure, the authors break the region into parts and reduce the continuous demand (our density) to a discrete demand. One difference between our work and the 1997 paper is that the population density (which is the demand in Drezner and Drezner) has nonuniform distribution. Another difference between the 1997 paper and the present paper is that Drezner and Drezner seek to locate one additional facility in a region already containing several competing facilities while the computations below aim for one, two, or three locations in a 'new' region. The authors in both the cited works then refer to the paper of Weiszfeld for convergence of the iterates. For our examples, begin with two locations along the line.

EXAMPLE 5 (two locations, one dimension: $n = 2$, $m = 1$). Consider the population distributed according to the tent-shaped piecewise linear function

$$\rho(x) = \begin{cases} 1+x & -1 \leqslant x < 0.5 \\ 3(1-x) & 0.5 \leqslant x \leqslant 1 \end{cases}$$

on the interval $[-1, 1]$. From Example 2a (and Theorem 3), we know that the optimal locations z_1 and z_2 are the centers of mass of the intervals $[-1, \frac{z_1+z_2}{2}]$ and $[\frac{z_1+z_2}{2}, 1]$. These centers of mass are given by $\frac{\int_{-1}^{(z_1+z_2)/2} x\rho(x)\,dx}{\int_{-1}^{(z_1+z_2)/2} \rho(x)\,dx}$ and $\frac{\int_{(z_1+z_2)/2}^{1} x\rho(x)\,dx}{\int_{(z_1+z_2)/2}^{1} \rho(x)\,dx}$. Because it is not known in advance where the midpoint $(z_1+z_2)/2$ lies relative to $x = 0.5$, it is awkward to compute z_1 and z_2 directly. Instead we can use fixed point iteration to approximate the solution. Beginning with a guess of $z_1 = 0$ and $z_2 = 1$, and using numerical approximations of integrals we obtain the following values approximating the solution.

iteration	z_1	z_2	total cost
0	0	1	0.1875
1	0	0.666666	0.119084
2	−0.111111	0.57075	0.0929902
3	−0.180135	0.519251	0.0843480
4	−0.220294	0.490973	0.0816279
5	−0.243107	0.475372	0.0807851
10	−0.270441	0.457092	0.0804116
15	−0.271811	0.456187	0.0804104
20	−0.271945	0.456099	0.0804104

A bit more analytic work (guessing that $z_2 < 0.5$) will give (to 6 decimal places) $z_1 = -0.271952$ and $z_2 = 0.456094$ with total cost 0.0804104. Of course, in general a fixed-point iteration scheme need not converge. Sufficient conditions for convergence are that the absolute values of appropriate partial derivatives are bounded above by 1. Checking this condition is usually not feasible analytically even for the examples considered here. However, the relevant functions can sometimes be graphed and it may be able to see that the graph satisfies the required bound. It should also be noted that unless additional information is known about uniqueness, it is not known when the fixed-point iteration will converge to the desired extreme values.

In the final three examples, we use Theorem 3 and iteration to approximate a solution. Since we have neither an analytic solution nor an independent proof of uniqueness of a solution, we cannot be assured that the sequences converge to an absolute minimum.

EXAMPLE 6 (three locations, one dimension: $n = 3$, $m = 1$). Let the density function be defined by $\rho(x) = x + 1$ on the interval $[-1, 1]$. Using the fact that the optimal locations are at the center of mass in each subregion, we need z_1, z_2, and z_3 to satisfy

$$z_1 = \frac{\int_{-1}^{(z_1+z_2)/2} x\rho(x)\,dx}{\int_{-1}^{(z_1+z_2)/2} \rho(x)\,dx}, \quad z_2 = \frac{\int_{(z_1+z_2)/2}^{(z_2+z_3)/2} x\rho(x)\,dx}{\int_{(z_1+z_2)/2}^{(z_2+z_3)/2} \rho(x)\,dx}, \quad z_3 = \frac{\int_{(z_2+z_3)/2}^{1} x\rho(x)\,dx}{\int_{(z_2+z_3)/2}^{1} \rho(x)\,dx}$$

Some results on the iterative search for a fixed point are shown in the table.

iteration	z_1	z_2	z_3	total cost
0	-1	0	1	0.166666
1	-0.666666	0.0833333	0.761904	0.0717035
2	-0.52777	0.105380	0.727542	0.0622800
3	-0.474132	0.132405	0.72482	0.0599563
4	-0.447274	0.155409	0.730181	0.0588423
5	-0.430611	0.173588	0.736428	0.0582013
10	-0.395361	0.217869	0.754094	0.0573511
15	-0.386354	0.229559	0.758913	0.0572912
20	-0.383974	0.232652	0.760191	0.0572870

EXAMPLE 7 (two locations, two dimensions: $n = 2$, $m = 2$). For the first of our two-dimensional examples, consider a population in the square $[-1, 1] \times [-1, 1]$ with density $\rho(x, y) = 1 + x + y^2$. The density is symmetric with respect to the x-axis and so we expect the optimal locations $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ to be symmetric also. Associated with each z_i there is a region D_i. The boundary between the regions is the perpendicular bisector of the line segment between z_1 and z_2 (this is because the cost function is the square of the l^2-norm). Thus once again the values x_1, y_1, x_2, and y_2 are given by the ratios of integrals over an appropriate region. The following table gives the values of the iterates in the search for a fixed point, beginning with the rather poor initial choice of $(-1, 1)$ and $(1, -1)$.

iteration	x_1	y_1	x_2	Y_2	total cost
0	-1	1	1	-1	7.11112
1	-0.0666667	0.533333	0.44	-0.32	2.20819
2	0.107753	0.601576	0.356231	-0.449263	1.96518
3	0.199947	0.599122	0.293035	-0.515121	1.90259
4	0.232314	0.58255	0.266425	-0.540998	1.89246
5	0.243367	0.572231	0.256405	-0.552551	1.8907
6	0.247416	0.567072	0.252542	-0.557893	1.89043

Since the density is a polynomial and since the iteration gives an idea of the location (and hence the form of the dividing line between the regions), it is possible (using Theorem 3) to calculate a location analytically for this example. The locations are (1/4, 9/16) and (1/4, - 9/16) with a total cost of $1381/720 \approx 1.89028$. These locations satisfy the necessary conditions for an optimal location given by Theorem 3. If these are the only such points, then they are optimal. Due to the various ways that a general dividing line between regions can be situated, a proof that this point is optimal is tedious.

EXAMPLE 8 (two locations, two dimensions: $n = 2$, $m = 2$). For the final example, consider a population in the square $[-1, 1] \times [-1, 1]$ with density

$$\rho(x, y) = \exp[-3(x - 0.5)^2 - 3(y - 0.25)^2].$$

When graphed, this function has a 'noticeable' maximum at (0.5, 0.25) and the density falls off quickly. The optimal locations are of the form $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Once again the values x_1, y_1, x_2, and y_2 are given by ratios of integrals over the appropriate region. The following table gives the values of several of the iterates in the search for a fixed point.

iteration	x_1	y_1	x_2	Y_2	total cost
0	-0.75	-0.75	0.5	0.25	0.225753
1	-0.25327	-0.329012	0.433098	0.236305	0.192400
2	-0.44077	-0.188259	0.478844	0.278479	0.172509
4	0.143155	-0.101715	0.534428	0.364032	0.157139
6	0.196772	-0.0934254	0.53799	0.400079	0.154001
8	0.274963	-0.9591	0.516551	0.454709	0.150142
10	0.328216	-0.10151	0.483896	0.484469	0.148184
12	0.362488	-0.104441	0.457407	0.497224	0.147359

Remarks on further results. The locations of facilities can be interpreted as the support of a measure. The value of the measure at each location is the mass of the region associated with the facility. In this paper, we put no restriction on the values of the measure. In the example of locating recycling centers, the centers may have limited capacity however. We have obtained results similar to those in this paper for the following cases: a facility has a prescribed capacity; a facility is 'located' in higher dimensional subsets (rather than point locations as above). These results will appear in McAsey and Mou (1999).

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